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$$s = \frac{(a+b)d'}{(a-b)(d'g-d)} \log \left[ \frac{1 - t^{1/\sqrt{2k(a-b)(d'g-d)}/\sqrt{(a+b)d'}}}{2} + \frac{1 - t^{1/\sqrt{2k(a-b)(d'g-d)}/\sqrt{(a+b)d'}}}{2} \right]$$

$$\text{or } s = \frac{(a+b)d'}{(a-b)(d'g-d)} \log(\cosh t^{1/\sqrt{2k(a-b)(d'g-d)}/\sqrt{(a+b)d'}}).$$

$$s = \frac{1}{p^2} \log(\cosh pt) \text{ where } p = \sqrt{\frac{2k(a-b)(d'g-d)}{(a+b)d'}}.$$

$$\text{For four seconds, } s = \frac{1}{p^2} \log(\cosh 4p).$$

[See Bowser's *Analytic Mechanics*, page 314, ex. 5, where  $v=0$  and  $d=0$ , of equation (3) above.]

Also solved by *ELMER SCHUYLER*.

## AVERAGE AND PROBABILITY.

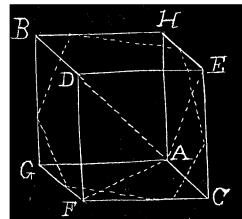
### 61. Proposed by COL. CLARKE.

A cube being cut at random by a plane, what is the chance that the section is a hexagon? [Erom Williamson's *Integral Calculus*.]

Solution by LEWIS NEIKIRK, Graduate Student, University of Colorado, Boulder, Col.

#### I. PRELIMINARY INVESTIGATION.

Let  $S$ , the random section, be determined by the coördinates  $p$ ,  $\varphi$ , and  $\theta$ ,  $\theta$  being the angle between  $p$  and its projection on  $ACFG$  and  $\varphi$  the angle between  $AC$  and the projection of  $p$ . Let  $P$  be the point of intersection of  $p$  and  $S$ . Also let  $p$  increase from zero for  $\varphi$  and  $\theta < \frac{1}{2}\pi$ .  $S$ , starting with three sides at  $A$ , gains three more, one at each of the corners  $C$ ,  $G$ , and  $H$ ; and loses three, one at each of the corners  $B$ ,  $E$ , and  $F$ .  $\varphi$  and  $\theta$  determine the order in which these gains and losses shall occur, and plainly  $S$  can be hexagonal only when the first loss is antedated by all three gains.



For  $p$  in the diagonal  $AD$  ( $\varphi = \frac{1}{2}\pi$ ,  $\theta = \cot^{-1}\sqrt{2}$ ), the three gains are simultaneous and are followed by three simultaneous losses. For  $p$  as an element of the area  $DAF$  ( $\varphi = \frac{1}{2}\pi$ ,  $\theta < \cot^{-1}\sqrt{2}$ ), one gain at  $H$  is followed by two more at  $C$  and  $G$ ; then two losses at  $B$  and  $E$ , followed by one loss at  $F$ . For  $p$  as an element of the areas  $DAE$  and  $DAB$  ( $\varphi < \frac{1}{2}\pi$  and  $\theta = \tan^{-1}\cos\varphi$ , and  $\varphi > \frac{1}{2}\pi$  and  $\theta = \tan^{-1}\sin\varphi$ ) there is a like sequence of gains and losses. So far it has been easy to enumerate the exact order in which all the gains and losses occur.

For  $p$  within the solid angle  $A-DECF$  ( $\varphi < \frac{1}{2}\pi$  and  $\theta < \tan^{-1}\cos\varphi$ ) the first loss occurs at  $B$ , and the last gain at  $C$ ; the order and place of the remaining gains and losses would be difficult to enumerate and is in any event immaterial

to the solution. A similar enumeration and statement to the angles  $A-DBGF$  and  $A-DBHE$ . Evidently, then, our attention may be confined to any one of these angles, say  $A-DECF$ .

## II. SOLUTION.

As  $p$  increases within the solid angle  $A-DECF$  ( $\varphi < \frac{1}{2}\pi$  and  $\theta < \tan^{-1}\cos\varphi$ ),  $S$ , starting at  $A$  with three sides, gains two sides successively at  $G$  and  $H$ , the order being immaterial; then for some values of  $\varphi$  and  $\theta$  it will gain a third side at  $C$  before losing the first one at  $B$ ; for other values this loss at  $B$  will antedate the third gain at  $C$ ; in fact it may antedate the second of the two gains at  $G$  or  $H$ . For certain values of  $\varphi$  and  $\theta$  between these extremes, this first loss and final gain will concur. In this last case  $p$  is an element of the area  $EAF$  ( $\varphi < \frac{1}{2}\pi$  and  $\theta = \tan^{-1}[\cos\varphi - \sin\varphi]$ ); for this area is plainly perpendicular to  $BC$ , and must therefore contain  $p$  when  $S$  reaches  $B$  and  $C$  simultaneously.

The number of cases which have the coördinates  $p$ ,  $\varphi$ , and  $\theta$  are  $dpd\omega = \cos\theta dpd\theta d\varphi$ , where  $\omega$  is a solid angle with its vertex at  $A$ . The integration of  $p$  for the favorable cases extends from  $p_1$  to  $p_2$ , where  $p_1$  is the value of  $p$  when  $S$  reaches  $C$ , and  $p_2$  is the value of  $p$  when  $S$  reaches  $B$ . The integration of  $\theta$  extends from the plane  $EAF$  to the plane  $DAE$ , and the integration of  $\varphi$  from 0 to  $\frac{1}{2}\pi$ . The above limits may be calculated from the following spherical triangles in which the primed letters refer to points on a unit sphere, center at  $A$ , corresponding to points with unprimed letters in the figure.

In the right spherical triangle  $P'C'F'$ ,  $P'C' = \psi_1$ ,  $F'C' = \varphi$ ,  $P'F' = \theta$ , and  $\angle P'F'C' = 90^\circ$ . Then  $p_1 = a\cos\psi_1 = a\cos\theta\cos\varphi$ , where  $a$  is an edge of the cube.

In the spherical triangle  $B'P'H'$ ,  $B'F' = 45^\circ$ ,  $B'P' = \psi_2$ ,  $P'F' = 90^\circ - \theta$ , and  $\angle B'H'P' = 90^\circ - \varphi$ . Then  $p_2 = a\sqrt{2}\cos\psi_2$ ,  $\cos\psi_2 = (1/\sqrt{2})(\sin\theta + \cos\theta\sin\varphi)$ .

$$\therefore p_2 = a(\sin\theta + \cos\theta\sin\varphi).$$

In the right spherical triangle  $F'P'C'$  right angled at  $C'$ ,  $P'C' = \theta_1$ ,  $F'C' = 45^\circ - \varphi$ , and  $\angle P'F'C' = \cot^{-1}(1/\sqrt{2})$ .

Then  $\tan\theta_1 = \sqrt{2}\sin(45^\circ - \varphi) = \cos\varphi - \sin\varphi$ . Then  $\theta_1 = \tan^{-1}(\cos\varphi - \sin\varphi)$ .

In the right spherical triangle  $P'H'E'$  right angled at  $E'$ ,  $H'E' = 45^\circ$ ,  $P'H' = 90^\circ - \theta_2$ , and  $\angle P'H'E' = \varphi$ . Then  $\tan\theta_2 = \cos\varphi$ ,  $\theta_2 = \tan^{-1}\cos\varphi$ .

All the favorable cases

$$F = 12 \int_0^{\frac{1}{2}\pi} \int_{\theta_1}^{\theta_2} \int_{p_1}^{p_2} \cos\theta dpd\theta d\varphi = 12a(\sqrt{3}\tan^{-1}\frac{1}{3}\sqrt{3} - \sqrt{2}\tan^{-1}\frac{1}{2}\sqrt{2}).$$

The integration for the total number of cases extends for  $p$  from 0 to  $p'$ , where  $p'$  is the value of  $p$  when  $S$  reaches  $D$ ; for  $\theta$ , from 0 to  $\frac{1}{2}\pi$ , and for  $\varphi$  from 0 to  $\frac{1}{2}\pi$ .

In the spherical triangles  $D'AP'$  and  $D'AF'$ ,  $D'A = \psi'$ ,  $P'A = \theta$ ,  $D'P' = \beta$ ,  $\angle D'AP' = \alpha$ ,  $F'A = (45^\circ - \varphi)$ ,  $F'D' = \tan^{-1}(1/\sqrt{2})$ ,  $\angle D'F'A = 90$ , and  $\angle D'AF' = (90^\circ - \alpha)$ .

Then  $\cos\psi' = \sqrt{\frac{2}{3}}\cos(45^\circ - \varphi) = (1/\sqrt{3})(\cos\varphi + \sin\varphi)$ ,  $\tan\alpha = \sqrt{2}\sin(45^\circ - \varphi)$   
 $= \cos\varphi - \sin\varphi$ ,

$$p = a\sqrt{3} \cos\beta = a\sqrt{3} (\cos\phi' \cos\theta + \sin\phi' \sin\theta \cos\alpha) \\ = a[\cos\theta(\cos\phi + \sin\phi) + \sin\theta].$$

The total number of cases is

$$T = 4 \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi} \cos\theta dp d\theta d\phi = 3\pi a.$$

Therefore the probability is  $P = E/T = 4/\pi (\sqrt{3} \tan^{-1} \frac{1}{3}\sqrt{3} - \sqrt{2} \tan^{-1} \frac{1}{2}\sqrt{2})$ .

NOTE.—I wish to acknowledge my indebtedness to Professor DeLong and Mr. Frank Giffin for looking over this solution and making valuable suggestions.

### MISCELLANEOUS.

70. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Prove  $\tan^{-1}x = \frac{1}{2i} \left( \log \frac{x-i}{x+i} \right)$ , and thence that  $\pi = (2/i) \log(i)$ .

I. Solution by GUY B. COLLIER and HAROLD C. FISKE, Class 1901, Union College, Schenectady, N. Y., and the PROPOSER.

Consider the integral,

$$\int \frac{dx}{1+x^2} = \tan^{-1}x \dots \dots (1).$$

Integrate the left member by partial fractions

$$\int \frac{dx}{1+x^2} = \frac{1}{2i} \int \frac{dx}{x-i} - \frac{1}{2i} \int \frac{dx}{x+i} = \frac{1}{2i} \log \frac{x-i}{x+i} \dots \dots (2).$$

$\therefore$  From (1) and (2),

$$\tan^{-1}x = \frac{1}{2i} \log \frac{x-i}{x+i}.$$

When  $x=1$  this becomes

$$\frac{1}{2} \pi = \frac{1}{2i} \log \frac{1-i}{1+i} = \frac{1}{2i} \log(-i).$$

$\pi = (2/i) \log(-i)$ , it should have been.

II. Solution by E. E. GAINES, A. M., Professor of Mathematics, Richmond College, Richmond, Va.

We have the identity

$$\frac{1}{1+x^2} = \frac{1}{2i} \left( -\frac{1}{i-x} - \frac{1}{i+x} \right). \therefore \int \frac{dx}{1+x^2} = \frac{1}{2i} \log \left( \frac{i-x}{i+x} \right) + c.$$